THE PLACE

OF

PARTIAL DIFFERENTIAL EQUATIONS

IN

MATHEMATICAL PHYSICS

BEING

A COURSE OF READERSHIP LECTURES
DELIVERED AT PATNA UNIVERSITY

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BY

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PREFACE.

THE lectures contained in this booklet were all delivered practically extempore. That they see the light of day, is chiefly due to the kindness of Mr. Gorakh Prasad, M.Sc., and Mr. Badrinath Prasad, M.Sc., Assistant Professors of Mathematics in the Benares Hindu University, of whom the former took down notes of the last three lectures and the latter of the first three. The copy for the press was prepared by me from these notes with the help of Mr. Mewa Ram Saksena, M.A., B.Sc., of the Shyamsundar Memorial Intermediate College, Chandausi. Dr. Bibhutibhusan Datta, D.Sc., Acting Head of the Department of Applied Mathematics in the University College of Science, Calcutta, kindly helped me in correcting the proof sheets. I take this opportunity to express my thanks to all these friends and former pupils.

GANESH PRASAD.

April, 1924.

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FIRST LECTURE.

[Delivered on the 10th January, 1921.]

INTRODUCTION.

The chief reason for my choosing "The place of partial differential equations in Mathematical Physics" as the subject for these lectures is my wish to inspire in my audience a love for Mathematics. Before entering into details, however, I shall give a brief historical account of the application of Mathematics to natural phenomena.

Mathematicians devote themselves to their subject chiefly with one of the two objects: either to study natural phenomena in connection with matter, space or time; oı for purely aesthetic pleasure. A good illustration of the latter is afforded by their attempt to find all the prime numbers less than a certain given number. In ancient times people studied Mathematics mainly because of some mystery

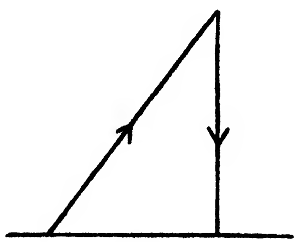

Fig. 1.

which they attached to it and not for its application in studying natural phenomena; in this connection, we may note, for example, the doctrines of Pythagoras or the *Sulvasutras* of Apastamba. Among the earliest writers who devoted themselves to Mathematics, were Archytas of Tarentum (428–327 B.C.), Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.). Archytas applied his knowledge of Mathematics in building up the science of Mechanics. After Archytas, Aristotle did something in this respect. But he had strange ideas about some elementary truths; for example, he supposed that for a falling body the rapidity of its motion depended upon its weight, according to him a heavy body falling more rapidly than a lighter one. Another erroneous idea which Aristotle had was that if a stone were thrown up at a certain angle, it

would go first upwards in a straight line at that angle and then fall vertically downwards, the path of the projectile consisting of two straight lines, as in the accompanying figure. Archimedes discovered the principle of the lever and the conditions of the floatation of a solid in a liquid. Galilei (1564-1642) was the first to acquire clear ideas about the first two laws of motion. He also stated and proved that the path of a projectile was not a pair of straight lines but a parabola. After Galilei, Mechanics became gradually a science. A large number of results were discovered between 1642 and 1700.

In the beginning of the 18th century a new epoch was created; for, people began to think of natural phenomena other than ordinary motions, such as for instance the state of a hot body. This gave rise to the use of partial differential equations. Let there be a solid body which conducts heat. The amount of heat passing

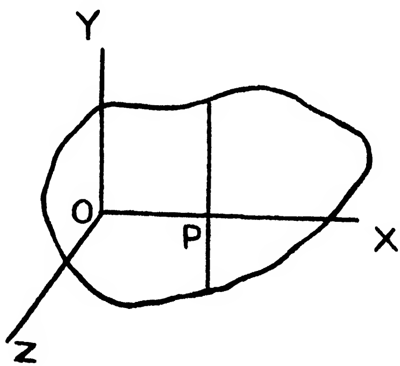

FIG. 2.

across a sheet at P perpendicular to the x-axis is required to be found out first. The amount of heat conducted will be proportional to the gradient of the temperature and is equal to

$$-k\frac{\partial V}{\partial x},$$

which is the first germ of truth in the Analytical Theory of Heat and was discovered about the year 1747; the partial differential equation governing the state of a hot body was discovered much later. Another important discovery was made in connection with the study of sound. People had already acquired some idea of harmony, and the formula of Taylor,

viz. $$\tau = 2l\sqrt{\frac{\rho}{T}},$$

was roughly known to Aristotle and the ancient *Rishis* of India. In 1636 Mersenne gave the above formula as the result of his experiments. But all such results were at first more or less of an empirical nature. The equation arising in the problem of the vibrating string, viz.

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

was given by D'Alembert as late as in 1747 and was the first partial differential equation to be carefully studied. It was solved in two ways. Putting $\frac{T}{\rho}=a^2$,

$$y=f(x+at)+F(x-at),$$

was the first solution given by Euler and D' Alembert. Another, which was given by Bernoulli, is

$$y=\sum_1^\infty A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

In appearance, Bernoulli's solution is periodic whereas the former solution is non-periodic. Therefore the two solutions formed the subject of a most interesting controversy. Fourier showed that Bernoulli's solution was also right and that, contrary to the views of Euler, it can be used for the case of an *arbitrary* displacement.

It is the deliberate opinion of modern authorities that before the use of partial differential equations, there was no sound basis upon which the science of Physics could rest. Soon after the time when partial differential equations came to be applied to the study of natural phenomena, Fourier made the following statement which will remain true for ever :—

"The analytical equations, unknown to the ancient geometers, which Descartes was the first to introduce into the study of curves and surfaces, are not restricted to the properties of figures, and to those properties which are the object of rational Mechanics; they extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, that is to say more worthy to express the invariable relations of natural things.

"Considered from this point of view, mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind.

"Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse,

and discovers the hidden analogies which unite them. If matter escapes us, as that of air and light, by its extreme tenuity, if bodies are placed far from us in the immensity of space, if man wishes to know the aspect of the heavens at successive epochs separated by a great number of centuries, if the actions of gravity and heat are exerted in the interior of the earth at depths which will be always inaccessible, mathematical analysis can yet lay hold of the laws of the phenomena. It makes them present and measurable, and seems to be a faculty of the human mind destined to supplement the shortness of life and the imperfection of the senses; and what is still more remarkable, it follows the same course in the study of all phenomena; it interprets them by the same language, as if to attest the unity and simplicity of the plan of the Universe, and to make still more evident that unchangeable order which presides over all natural causes."

We have the following important partial differential equations that occur in the study of natural phenomena:—

(1) $\dfrac{\partial^2 V}{\partial t^2} = c^2 \dfrac{\partial^2 V}{\partial x^2}$.

(2) $\dfrac{\partial V}{\partial t} = c^2 \left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}\right)$,

(3) $\dfrac{\partial^2 V}{\partial t^2} = c^2 \left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}\right)$.

(4) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$.

(5) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} + \lambda^2 V = 0$.

(6) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = -4\pi\rho$.

(7) $A\dfrac{\partial^2 V}{\partial t^2} + B\dfrac{\partial V}{\partial t} = \dfrac{\partial^2 V}{\partial x^2}$.

As an illustration of the development of the rigorous method for the study of natural phenomena, I will sketch an outline of the history of the proof of Poisson's equation,

$$\nabla^2 V = -4\pi\rho.$$

The proof of Poisson's equation, usually given in ordinary text-books, is not quite correct. The proof is based on enclosing the point (x, y, z) in a small sphere where the density ρ is taken to be constant. $V = V_1 + V_2$ where V_2 is the potential due to the small sphere and V_1 is the potential due to the rest of the attracting body. $\nabla^2 V_1 = 0$ and thus $\nabla^2 V = \nabla^2 V_2$. Now if you postulate that ρ is constant everywhere in the small sphere, then of course Poisson's

proof can hold true. But in case ρ is variable, this proof will no longer hold. Green took up this question in 1828 and said that ρ is not constant but its variation is very small in the sphere and is therefore negligible. The first rigorous proof was published in 1840 by Gauss who took ρ and its first partial differential coefficients to be continuous. Now suppose V, the potential at $P(x, y, z)$, is due to a solid for which ρ at any point Q is

$$\frac{\cos^2\theta}{\log\frac{1}{r}},$$

where $PQ=r$, the radius vector and θ is the angle made by the radius vector with the axis of x. In this case Poisson's equation is not valid at P; for,

$$\frac{\partial^2 V}{\partial x^2}=\infty, \quad \frac{\partial^2 V}{\partial y^2}=-\infty, \quad \frac{\partial^2 V}{\partial z^2}=-\infty \text{ and } \infty-\infty-\infty$$

has no meaning. I invite your attention to Professor Petrini's generalization of Poisson's equation, viz.

$$\underset{\substack{h=0,\\k=0,\\l=0,}}{Lt}\left[\frac{1}{h}\left\{\frac{\partial V(x+h, y, z)}{\partial x}-\frac{\partial V}{\partial x}\right\}+\frac{1}{k}\left\{\frac{\partial V(x, y+k, z)}{\partial y}-\frac{\partial V}{\partial y}\right\}+\frac{1}{l}\left\{\frac{\partial V(x, y, z+l)}{\partial z}-\frac{\partial V}{\partial z}\right\}\right]=-4\pi\rho,$$

where h, k, and l tend to zero without the ratio of any two of them tending to zero or infinity. There are various advantages of this generalization, but in my paper[1] "On the failure of Poisson's equation and of Petrini's generalization," I indicated a case in which this generalization fails. Take for example

$$\rho=\cos\log\frac{1}{r};$$

then there is no limit in this case except for very special values of h, k, l, and the generalization as well as Poisson's equation fails.

The application of the modern Theory of Functions of a real variable makes the use of partial differential equations more logical. Also the application ot the Theory of Integral Equations is very useful. A typical integral equation is

$$\phi(x)=f(x)+\lambda\int_a^b K(x, \xi)\phi(\xi)d\xi,$$

[1] *Bulletin of the Calcutta Mathematical Society*, Vol. 8, 1918.

where ϕ is the unknown function to be determined. The first important paper which laid the foundation of the Theory of Integral Equations as a systematic branch of Mathematics was published by Professor Ivar Fredholm in 1900 and was later published in the *Acta Mathematica* in an extended form in 1903. Integro-differential equations of a somewhat complicated type are useful in the "Mechanics of Heredity," i.e. those branches of Mechanics in which the state at a particular point of time is dependent not only on the immediate past but also on the whole period antecedent to that point of time; as for example in the phenomena of Hysteresis. Integro-differential equations are equations which involve not only the unknown functions under signs of integration but also the unknown functions themselves and their derivates. A paper on the subject of integro-differential equations by Professor Volterra appeared in the *Acta Mathematica*, Vol. 35. The subject of integral equations is of essential importance in studying the phenomena of

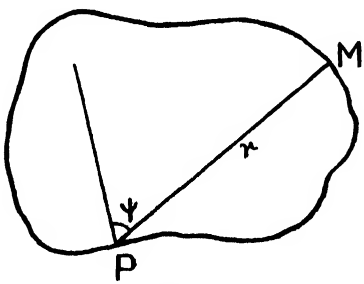

FIG. 3.

radiation. The law of Kirchoff, $\frac{\eta}{a}=$ constant, cannot be proved without the help of integral equations. The first rigorous proof was given by Professor David Hilbert in 1913. The kinetic theory of gases cannot be rigorously studied without the help of integral equations (see Hilbert's paper in the *Mathematische Annalen*, Vol. 72, 1912).

There can be no better instrument for studying natural phenomena than integral equations, partial and ordinary, and integro-differential equations. The truth of this remark can be easily seen by considering the problem of Dirichlet: To find a solution of $\nabla^2 V=0$ inside a given region (Fig. 3) when, on the surface bounding the region, $V=f(x, y, z)$, a prescribed function. By the help of the integral equation

$$\rho_{(M)}=f(M)+\lambda\int\frac{\cos\psi}{r^2}\;\rho_{(P)}\;ds,$$

the quantity ρ can be found as the strength of a double layer whose potential is the required solution of the problem. Integral equations are also most serviceable in studying the problem of the conduction of heat in a body. The well-known partial differential equation for the linear conduction of heat is

$$\frac{\partial V}{\partial t}=c^2\frac{\partial^2 V}{\partial x^2}.$$

But this equation is invalid for some very simple cases and has to make way for a partial integral equation. (See G. Prasad's memoir "Constitution of matter and analytical theories of heat" in the *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 1903).

SECOND LECTURE.

[Delivered on the 11th January, 1921.]

THE PARTIAL DIFFERENTIAL EQUATIONS OF VIBRATORY PHENOMENA.

The most important partial differential equation met with in the treatment of ordinary vibratory phenomena is of the form

$$\frac{\partial^2 V}{\partial t^2}=c^2\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right).$$

Vibratory phenomena of electrodynamic origin are governed by the equations of electrodynamics :—

$$4\pi u=\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},$$

$$4\pi v=\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},$$

$$4\pi w=\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y},$$

$$-\frac{\partial a}{\partial t}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},$$

$$-\frac{\partial b}{\partial t}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},$$

and

$$-\frac{\partial c}{\partial t}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y},$$

where the components of magnetic force at the point (x, y, z) are denoted by α, β, γ, the components of magnetic induction by a, b, c, the components of electric force by X, Y, Z, and the components of total current strength by u, v, w. These six equations represent, in symbols, the laws of Faraday, and their solution reduces to the solution of a single partial differential equation of the type

$$\frac{\partial^2 F}{\partial t^2}=c^2\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}\right)+4\pi c^2 u .$$

In the study of many kinds of elastic vibrations, we come across a partial differential equation of a similar type. In the case of the vibrations of an elastic plate, the motion is given by solving an equation of the form

$$\frac{\partial^2 w}{\partial t^2} = \nabla^2(\nabla^2 w),$$

where, as usual, ∇^2 stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

I proceed to give you two examples to show how partial differential equations are utilized to determine the vibrations in concrete cases :—

First consider the motion of air in a rigid spherical envelope. Transforming the equation

$$\frac{\partial^2 V}{\partial t^2} = c^2 \nabla^2 V$$

into polar co-ordinates, we have

$$\frac{\partial^2 V}{\partial t^2} = c^2 \frac{1}{r^2} \left\{ \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \right\}.$$

Putting $V = U(r, \theta, \phi) \cdot \cos kct$, we find that the equation is satisfied by

$$U = \frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} S_n(\theta, \phi),$$

where $S_n(\theta, \phi)$ is any surface harmonic of degree n. Certain boundary conditions are to be satisfied in every case. Here

$$\frac{J_{n+\frac{1}{2}}(ka)}{\sqrt{a}} = 0,$$

a being the radius of the envelope. This gives the suitable values of k and thus we have an infinite number of solutions each satisfying the differential equation and the condition at the boundary. The sum of such solutions

$$V = \sum_{n,\lambda} \frac{J_{n+\frac{1}{2}}(k_\lambda r)}{\sqrt{r}} S_{n,\lambda}(\theta, \phi) \cos k_\lambda ct$$

must also satisfy the differential equation and the boundary condition and gives the motion, the constants in $S_{n,\lambda}$ being so chosen as to give the prescribed initial state when t is equated to zero.

As a second example, consider the vibrations of a circular membrane.

$$\frac{\partial^2 V}{\partial t^2} = c^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

is the equation of motion. On transforming it into polar co-ordinates, we have

$$\frac{\partial^2 V}{\partial t^2} = c^2\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}\right).$$

Proceeding as in the first example,

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+k^2U=0.$$

The typical solution of this equation is

$$U=J_n\,(kr)\,(A_n \cos n\,\theta + B_n \sin n\,\theta).$$

Since at the boundary $r=a$, there is no displacement, U is zero there and therefore k is a root of the equation

$$J_n(ka)=0.$$

The motion is given by

$$V=\sum_{n,\lambda} J_n(k_\lambda r) \cos k_\lambda ct\,(A_{n,\lambda} \cos n\theta + B_{n,\lambda} \sin n\theta),$$

the constants being determined by the prescribed initial state.

Now I want to give you some criticisms, from the point of view of the theory of functions of real variables, on the usual method adopted for the study of vibratory phenomena. For the sake of simplicity and fixity of ideas, I will consider the simplest vibratory phenomenon, viz. the vibrations of a string.

The partial differential equation of the motion is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2},$$

where c^2 stands for $\frac{T}{\rho}$ and may be taken to be unity if the units are suitably chosen. Let the string be fixed at its extremities $x=o$, $x=\pi$ and let it be set in motion by a displacement $y=f(x)$. Then the motion at any subsequent time t is given by

$$y=\sum_{n=1}^{\infty} A_n \sin nx \cos nt\,,$$

the constants being chosen in such a way as to satisfy the initial condition. Therefore

$$f(x)=\sum_{1}^{\infty} A_n \sin nx$$

and

$$A_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.$$

The flaws in the above method for determining the motion of the vibrating string were pointed out by me in a paper[1] "On the

[1] See Vol. 6 of the "*Bulletin of the Calcutta Mathematical Society*," 1916.

ibrating string with an infinite number of edges." The following re the two flaws :—(1) We always assume the existence of

$$\frac{\partial^2 y}{\partial t^2} \text{ and } \frac{\partial^2 y}{\partial x^2}$$

t every point. (2) It is taken for granted that if each term of an ıfinite series satisfies the differential equation then the sum of the eries will also satisfy the equation.

Both the assumptions are unjustifiable. For, a string may ave an edge so that at that point

$$\frac{\partial y}{\partial x}, \text{ what to speak of } \frac{\partial^2 y}{\partial x^2},$$

rill be non-existent; also what is true of a finite number of terms ıay not be true of an infinite number of terms. These objections proceed to elucidate at some length.

Consider the string which is under constant tension T. Acording to Newton's second law, study the change of momentum f a small element of the string in the period of time

$$(t_1 - \tau,\ t_1 + \tau).$$

et the x-co-ordinates of the ends of the element be $x_1 - h$, $x_1 + h$. 'hen the change in its momentum is

$$\rho \int_{x_1 - h}^{x_1 + h} \left\{ \left(\frac{\partial y}{\partial t}\right)_{t = t_1 + \tau} - \left(\frac{\partial y}{\partial t}\right)_{t = t_1 - \tau} \right\} dx.$$

It must be balanced by the change in the force which is

$$T \int_{t_1 - \tau}^{t_1 + \tau} \left\{ \left(\frac{\partial y}{\partial x}\right)_{x = x_1 + h} - \left(\frac{\partial y}{\partial x}\right)_{x = x_1 - h} \right\} dt. \tag{1}$$

)nly the assertion of this equality gives the true equation of the)henomenon as an integro-differential equation.

What is usually done is to consider $2\,h$ as getting smaller and maller and also $2\,\tau$ getting smaller and smaller. Then the appliation of Rolle's theorem shows that the integro-differential equaion is equivalent to

$$2h \,.\, 2\tau \,.\, \left(\frac{\partial^2 y}{\partial t^2}\right)_{\substack{t = t_1 + \theta\tau, \\ x = x_1 + \theta_1 h}} = 2\tau \,.\, 2h \,.\, \frac{T}{\rho} \left(\frac{\partial^2 y}{\partial x^2}\right)_{\substack{t = t_1 + \theta_2 \tau, \\ x = x_1 + \theta_3 h}}$$

vhere the θ's are all proper fractions.

The conclusion, that

$$\left(\frac{\partial^2 y}{\partial t^2}\right)_{\substack{x=x_1,\\ t=t_1}} = \frac{T}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right)_{\substack{x=x_1,\\ t=t_1}},$$

is not always right; if there is discontinuity then the above equation may be invalid.

Now I proceed to elucidate the second flaw by taking the initial state of the string to be $y=f(x)$, where

$$f(x)=x \text{ for } 0 \leqq x \leqq \frac{\pi}{2}, \text{ and } f(x)=\pi - x \text{ for } \frac{\pi}{2} \leqq x \leqq \pi.$$

The ordinary procedure gives the displacement y at any time as

$$\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)x\cos(2n-1)t}{(2n-1)^2},$$

the assumption being that

$$\frac{\partial^2 y}{\partial t^2} = -\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sin(2n-1)x\cos(2n-1)t$$

and

$$\frac{\partial^2 y}{\partial x^2} = -\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sin(2n-1)x\cos(2n-1)t$$

and therefore the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

is satisfied. But the assumptions are unjustified because the series to which

$$\frac{\partial^2 y}{\partial t^2} \text{ as well as } \frac{\partial^2 y}{\partial x^2}$$

is equated *has no meaning, it being not convergent.* Thus both

$$\frac{\partial^2 y}{\partial t^2} \text{ and } \frac{\partial^2 y}{\partial x^2}$$

have to be investigated by a different procedure and their equality established by a different method.

How to study then the phenomenon of the vibrating string? This question was answered correctly by the late Dr. A. Harnack[1] in 1887. If

$$y(x,t),\ \frac{\partial^2 y}{\partial x^2} \text{ and } \frac{\partial^2 y}{\partial t^2}$$

---

[1] See his paper, "Ueber die mit Ecken behafteten Schwingungen gespannter Saiten" (*Mathematische Annalen*, Bd. 29).

are finite and continuous then it was proved by Harnack that the most general motion of the string is given by

$$y = \sum_{n=1}^{\infty} A_n \sin nx \cos nt.$$

Proof :—

Since $\frac{\partial y}{\partial x}$ is existent and finite, according to Fourier's theorem $y(x, t)$ must be equal to

$$\sum_{n=1}^{\infty} a_n(t) \sin nx,$$

where
$$a_n = \frac{2}{\pi} \int_0^{\pi} y(x, t) \sin nx \, dx.$$

[Here it may be noted that every continuous function is not expansible in Fourier's series. This was first shown by Du Bois Reymond who gave an example of a continuous function the Fourier's series corresponding to which is *not* convergent.]

Integrating by parts

$$\int_0^{\pi} \frac{\partial^2 y}{\partial x^2} \sin nx \, dx = -n^2 \int_0^{\pi} y \sin nx \, dx = -n^2 \frac{\pi}{2} a_n(t).$$

Since $\frac{\partial^2 y}{\partial t^2}$ is continuous, we have

$$\frac{\partial}{\partial t} \int_0^{\pi} y \sin nx \, dx = \int_0^{\pi} \frac{\partial y}{\partial t} \sin nx \, dx,$$

which equality is true because $\frac{\partial y}{\partial t}$ is continuous. Therefore similarly

$$\frac{\partial^2}{\partial t^2} \int_0^{\pi} y \sin nx \, dx = \int_0^{\pi} \frac{\partial^2 y}{\partial t^2} \sin nx \, dx;$$

and this latter integral equals

$$\int_0^{\pi} \frac{\partial^2 y}{\partial x^2} \sin nx \, dx,$$

because
$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

Hence
$$\frac{\partial^2}{\partial t^2} a_n + n^2 a_n = 0.$$

Therefore $a_n = A_n \cos nt + B_n \sin nt.$

If the string is simply displaced initially, then from

$$y = \sum_{n=o}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx$$

we find that the B's must be all zero.

Therefore the general solution is

$$y = \sum_{1}^{\infty} A_n \cos nt \sin nx.$$

I proceed to consider the case when there is an edge at a point P on the string so that at that point $\frac{\partial y}{\partial x}$ is non-existent and consequently $\frac{\partial^2 y}{\partial x^2}$ is also non-existent there.

Denote the ordinates by y_1, y_2 in the portion APB and by y'_1, y'_2 in the portion $AP'B$. The usual phenomenon of wave motion is that every wave propagates itself. Consider the small element of the string round P during a small period of time τ. This

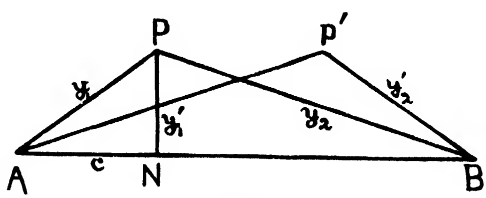

FIG. 4.

element is $\frac{dc}{dt}\tau$ where c is the abscissa of P; the element undergoes a change

$$\rho \frac{dc}{dt} \tau \left\{ \frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} \right\}$$

in momentum and the change is brought about by the impulse

$$T \left\{ \frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x} \right\},$$

acting for the period of time τ. Therefore, since $\frac{T}{\rho} = 1$, the units being suitably chosen, at P

$$\frac{dc}{dt} \left\{ \frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} \right\} = \frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}. \qquad (2)$$

As the string is continuous $y_1(c, t) = y_2(c, t)$; therefore

$$\frac{\partial y_1}{\partial c} \cdot \frac{dc}{dt} + \frac{\partial y_1}{\partial t} = \frac{\partial y_2}{\partial c} \cdot \frac{dc}{dt} + \frac{\partial y_2}{\partial t}.$$

This gives
$$\frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} = \frac{dc}{dt}\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right) \tag{3}$$

In order that the two equations (2) and (3) be consistent, $\frac{dc}{dt}$ must be equal to 1. Therefore the equation, which must be satisfied at the edge in the place of the partial differential equation, which has no meaning there, is

$$\left(\frac{\partial y}{\partial t}\right)_{c-0} - \left(\frac{\partial y}{\partial t}\right)_{c+0} = \left(\frac{\partial y}{\partial x}\right)_{c+0} - \left(\frac{\partial y}{\partial x}\right)_{c-0}.$$

The question is now whether in the present case the general solution holds; it will be easily seen that the required condition at the edge is satisfied by the solution and therefore the answer to the question is in the affirmative.

Cases of strings with three or four or any *finite* number of edges have been considered by Harnack. But the case of an *infinite* number of edges was treated for the first time by me in the paper already referred to. As a simple example, it is proved there that

$$y = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{C_n}{n^2} \cos nt \sin nx$$

gives the motion of the string, vibrating with the *infinite* number of edges, the abscissæ of which initially formed the set

$$\left\{ \frac{\pi}{2} - \frac{\pi}{2^{i+1}} \right\};$$

C_n being equal to

$$(-1)^{\frac{n}{2}-1} \sum_{i=1}^{\infty} \frac{1}{2^i} \sin \frac{n\pi}{2^{i+1}} \text{ or } (-1)^{\frac{n-1}{2}} \sum_{i=1}^{\infty} \frac{1}{2^i} \cos \frac{n\pi}{2^i},$$

according as n is even or odd.

I will conclude this lecture by showing how the theory of integral equations may be applied in the study of the problem of the motion of a vibrating string.

Let us consider the case of a string of *variable* density. Then in the equation

$$\rho(x) \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \tag{4}$$

ρ is not constant. Lord Rayleigh could not integrate this equation except for very special types of ρ.

The general solution can be obtained by the help of an integral equation. For, put

$$y = \cos nt \, . \, U(x).$$

Then the equation (4) becomes

$$\frac{\partial^2 U}{\partial x^2} + \frac{n^2}{T} \rho(x) \, U = 0,$$

and it can be shown without difficulty that this last equation together with the boundary conditions

$$U(0) = U(\pi) = 0$$

is equivalent to the integral equation

$$U(x) = \frac{n^2}{T} \int_0^\pi G(\xi, x) \, \rho(\xi) \, U(\xi) \, d\xi,$$

where
$$G(\xi, x) = \frac{(\pi - x)\,\xi}{\pi} \text{ for } \xi \leqq x,$$

$$G(\xi, x) = \frac{(\pi - \xi)\,x}{\pi} \text{ for } \xi \geqq x.$$

THIRD LECTURE.

[Delivered on the 12th. January, 1921.]

THE PARTIAL DIFFERENTIAL EQUATIONS OF THE ANALYTICAL THEORY OF THE CONDUCTION OF HEAT.

After the discovery of the experimental foundations of the elementary laws of conduction and radiation by Newton, Lambert and Biot, a good deal of time elapsed before the discovery by Fourier of the following differential equation as representing the phenomenon of the conduction of heat in an isotropic solid :—

$$\frac{\partial V}{\partial t} = c^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right),$$

where c^2 is the coefficient of diffusivity. A number of special problems were solved by Fourier and his contemporary Poisson. For example, take the problem of the cooling of a solid sphere, whose surface is maintained at a constant temperature, say zero : the initial state being given by

$$V(x, y, z, 0) = f(x, y, z) .$$

The particular case in which $f(x, y, z)$ is a function of r alone was solved by Fourier; the general case was considered by Poisson. To determine the state inside the sphere at time t, we begin by transforming the partial differential equation into polar co-ordinates. Thus

$$\frac{\partial V}{\partial t} = c^2 \left[\frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} \right\} \right].$$

Putting

$$V = e^{-c^2k^2t}\, U(r, \theta, \phi),$$

we have

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 U}{\partial \phi^2} + r^2k^2U = 0$$

of which the typical solution is

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\, S_n(\theta, \phi)$$

where $S_n(\theta, \phi)$ is a spherical surface harmonic of degree n. At the boundary $r = a$, V is zero ; therefore k must satisfy

$$J_{n+\frac{1}{2}}(ka) = 0 .$$

Let the values of k be k_1, k_2, k_3, etc. Then the required solution of the problem is given by

$$V = \sum_{m,\,n} \frac{J_{n+\frac{1}{2}}(k_m r)}{\sqrt{r}} S_{n,\,m}(\theta,\phi)\, e^{-k_m^2 c^2 t}$$

the summation extending over all values of n from 0 upwards and over all values of m from 1 upwards and the constants in $S_{n,\,m}$ being so chosen as to satisfy the condition relating to the initial state.

As another example, consider the state of heat of an ellipsoid of three unequal axes. Lamé in 1837, Heine in 1839 and Mathieu in 1870 made important contributions to the advancement of the knowledge of the subject. Transforming the partial differential equation $\nabla^2 V = 0$ into ellipsoidal co-ordinates, the equation reduces to

$$\left\{ (\mu-\nu)\left(R_\lambda \frac{\partial}{\partial\lambda}\right)^2 V + (\nu-\lambda)\left(R_\mu \frac{\partial}{\partial\mu}\right)^2 V + (\lambda-\mu)\left(R_\nu \frac{\partial}{\partial\nu}\right)^2 V \right\} = 0$$

where R_θ stands for $\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}$. Taking V to be of the form $E(\lambda)\,E(\mu)\,E(\nu)$ where $E(\theta)$ is a function to be determined, the equation will be satisfied if

$$\left(R_\theta \frac{\partial}{\partial\theta}\right)^2 E(\theta) = A\theta + B,$$

where A and B are constants. The function $E(\theta)$ is called Lamé's function and the product $E(\lambda)\,E(\mu)\,E(\nu)$ is called an ellipsoidal harmonic. The stationary state of heat of the ellipsoid can be expressed in terms of a series of such products. As regards the non-stationary state of heat, the partial differential equation of the phenomenon is

$$\frac{\partial V}{\partial t} = c^2 \nabla^2 V$$

and this reduces to the form

$$c^2 M \left\{ (\mu-\nu)\left(R_\lambda \frac{\partial}{\partial\lambda}\right)^2 V + (\nu-\lambda)\left(R_\mu \frac{\partial}{\partial\mu}\right)^2 V + (\lambda-\mu)\left(R_\nu \frac{\partial}{\partial\nu}\right)^2 V \right\} = \frac{\partial V}{\partial t}$$

where

$$M = \frac{-4}{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}.$$

Mathieu made an unsuccessful attempt to solve the above equation. For the special case of an ellipsoid of revolution, Professor C. Niven's results are of some interest. The most successful

attempt to solve the problem of the non-stationary state that has been made up to now is that of Mr. Bibhutibhuson Datta, who obtained, for the general case of the ellipsoid of three unequal axes as well as for the particular case of the ellipsoid of revolution, approximate results of very great interest and importance, one of the results being the correction of a mistake of Prof. Niven. (See the *Bulletin of the Calcutta Mathematical Society*, Vol. 8, and the *American Journal of Mathematics*, Vol. 41).

I propose now to submit the use of partial differential equations to a criticism from the point of view of one who wishes to be strictly rigorous. For the sake of simplicity and fixity of ideas, I will consider the conduction of heat in an infinite slab bounded by the faces $x=-\pi$, $x=\pi$, which are impermeable to heat, and I will take the initial temperature of the slab to be $f(x)$ where $f(x)$ is an even function of x.

The partial differential equation of the phenomenon is

$$\frac{\partial V}{\partial t}=c^2\frac{\partial^2 V}{\partial x^2},$$

where c^2 stands for the coefficient of diffusivity and may be taken to be unity if the units are suitably chosen. Also because of the impermeability of the bounding faces, $x=\pi$, $x=\pi$, we have $\frac{\partial V}{\partial x}=0$ for $x=\pi$ as well as for $x=-\pi$. Thus the state of the slab is given by

$$V=\tfrac{1}{2}a_0+\sum_1^{\infty} a_n e^{-n^2t}\cos nx,\ t\geqq 0, -\pi\leqq x\leqq\pi.$$

where

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

The above-mentioned treatment of the problem is the orthodox one and will be found in every text-book. In Fourier's original papers the statement of the conditions is the same as above, with this difference that it is clearly mentioned that

$$\frac{\partial V}{\partial t}=\frac{\partial^2 V}{\partial x^2}$$

for every positive value of t including zero and for every value of x in the interval $(-\pi, \pi)$.

In order to establish the defective character of the orthodox treatment, I take the simple case in which $f(x)$ is given by the accompanying graph.

At O there is no differential coefficient and still by the orthodox method

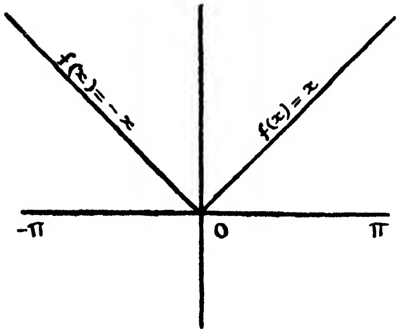

FIG. 5.

$$V = \frac{\pi}{2} - \frac{4}{\pi} \sum_{1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} e^{-(2n-1)^2 t}$$

If we differentiate the above series term by term the series so obtained is uniformly convergent as long as $t > 0$ and hence we can differentiate in that manner for such values of t. Thus for $t > 0$,

$$\frac{\partial V(0, t)}{\partial t} = \sum_{1}^{\infty} e^{-(2n-1)^2 t}$$

and, making t tend to 0,

$$\lim_{t=0} \frac{\partial V(0, t)}{\partial t} = \infty .$$

Now according to the theory of functions of real variables, if

$$\lim_{t=0} \frac{\partial V(0, t)}{\partial t}$$

is existent, it must be equal to

$$\lim_{t=0} \frac{V(0, t) - V(0, 0)}{t} .$$

Therefore, at the time $t=0$ the value of $\frac{\partial V}{\partial t}$ at O is infinite and also at that point, there being no differential coefficient of V with respect to x, $\frac{\partial^2 V}{\partial x^2}$ is non-existent; consequently for $x=0$, $t=0$, the equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$$

has no meaning and the orthodox treatment fails.

How to study then the phenomenon of the conduction of heat?

This question was answered correctly for the first time in my memoir, "Constitution of matter and analytical theories of heat". published by the Royal Society of Sciences of Göttingen in 1903. (*See Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Bd. 2, No. 1). In the theory, as remodelled by me, the equation of the phenomenon is shown to be not a partial differential equation but a partial integral equation, and attention is drawn to the simple principle that it is physical continuity

which is necessary and not the continuity of the form of an analytical expression. I proceed to elucidate these features of the theory.

First, as regards the equation of the phenomenon, this follows from the application of the principle of the conservation of energy in accordance with which the amount of heat passing into a volume must be equal to the amount of heat absorbed to raise the temperature of the mass of the volume. In the form of symbols, this condition may be written as :—

$$(A) \qquad \int_0^t \frac{\partial T(x, t')}{\partial x}\, dt' = \int_{-\pi}^{x} \left\{ T(x', t) - T(x', 0) \right\} dx', \ (-\pi \angle x \angle \pi),$$

$T(x, t)$ denoting the temperature. It can be easily shown that this integral equation is equivalent to the differential equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

if $\dfrac{\partial T}{\partial t}$ and $\dfrac{\partial^2 T}{\partial x^2}$ are finite and continuous. For, by the mean value theorem,

$$\int_{t_1}^{t_2} \left\{ \frac{\partial T(x_2, t')}{\partial x_2} - \frac{\partial T(x_1, t')}{\partial x_1} \right\} dt' = (t_2 - t_1) \cdot (x_2 - x_1) \cdot \left[\frac{\partial^2 T(x, t)}{\partial x^2} \right]_{\substack{x = x_1 + \theta_1 (x_2 - x_1) \\ t = t_1 + \theta_2 (t_2 - t_1)}},$$

θ_1 and θ_2 being proper fractions. Similarly

$$\int_{x_1}^{x_2} \left\{ T(x', t_2) - T(x', t_1) \right\} dx' = (x_2 - x_1)\ (t_2 - t_1) \left[\frac{\partial T(x, t)}{\partial t} \right]_{\substack{x = x_1 + \theta_3 (x_2 - x_1), \\ t = t_1 + \theta_4 (t_2 - t_1).}}$$

From (A), the two integrals given above are equal; consequently, proceeding to the limit by making $t_2 - t_1$ and $x_2 - x_1$ tend to zero, we have

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

for $x = x_1$, $t = t_1$, if these differential coefficients are finite and continuous.

Secondly, $T(x, 0)$ need not be derivable from the analytical expression, which equals $T(x, t)$ when $t > 0$, by putting $t = 0$ in that

expression. Thus, in the remodelled theory the convergence of the Fourier's series

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos nx$$

has no place. On the other hand the continuity of $T(x, t)$, i.e., the equality of

$$\lim_{t=+0} V(x, t) \text{ to } f(x),$$

follows at once from the continuity of $f(x)$; and the only restriction imposed on $f(x)$, in order that $V(x, t)$ should be the solution, is the following:—

(B) For each value of x, within the interval $(-\pi, \pi)$, the upper limit of $\left|\dfrac{\partial V}{\partial x}\right|$ in the neighbourhood of $t=0$ is finite.

The necessity for the finiteness of $\dfrac{\partial V}{\partial x}$ is obvious; for, since $-k\dfrac{\partial V}{\partial x}$ is the amount of heat which flows across an unit area of the plane at x in unit time, if this amount becomes infinite it will be absurd. Now for $t>0$, the series for $V(x, t)$ being uniformly convergent and the series obtained by differentiating it being also uniformly convergent, $\dfrac{\partial V}{\partial x}$ is finite. Hence we have to consider only the neighbourhood of $t=0$.

The equation (A) is satisfied by

$$T(x, t) = V(x, t), \; t>0$$

as long as $f(x)$ is finite and integrable. For, for $t>0$, the series for $\dfrac{\partial V}{\partial x}$ and V can be integrated term by term, the series concerned being all uniformly convergent.

As regards (B), it can be satisfied even if $\dfrac{\partial V}{\partial t}$ makes an infinite number of oscillations as t tends to 0; for example, if

$$f(x) = \int_0^x \cos\left(\log\frac{1}{x^2}\right) dx,$$

$\dfrac{\partial V}{\partial t}$ makes an infinite number of oscillations with indefinitely increasing amplitudes as t tends to zero. Thus theoretically the slab may, at certain points, become hotter and cooler an infinite number of times.

I conclude this lecture by pointing out that a number of interesting problems in the theory of the conduction of heat can be best solved by the use of the theory of linear integral equations. For example, one of the earliest applications of that theory is by Professor Picard, to the problem of the cooling of a ring. The partial differential equation of the phenomenon is

$$\frac{\partial V}{\partial t} = c^2 \cdot \frac{\partial^2 V}{\partial x^2} - b^2 V.$$

This differential equation can be solved in terms of the solutions of an ordinary differential equation of the second order which, in turn, is equivalent to an integral equation of the type

$$U(\xi) = F(\xi) + \lambda \int U(x)\, A(x)\, G(x, \xi) dx\,,$$

where U is the unknown function.

FOURTH LECTURE.

[Delivered on the 3rd February, 1921.]

THE PARTIAL DIFFERENTIAL EQUATIONS OF THE THEORY OF GRAVITATIONAL ATTRACTIONS AND THOSE OF ELECTROSTATICS AND MAGNETOSTATICS.

In the three lectures which I delivered last month, I first gave a historical review of the present theory of the partial differential equations of Mathematical Physics together with some criticisms and then dealt in succession with the equations of vibratory phenomena and of the analytical theory of the conduction of heat.

I wish to preface the three lectures which I intend to deliver in this month by explaining to you the two kinds of Mathematical Physics. This cannot be better done than in the following words addressed by Professor Vito Volterra of Rome at the inauguration of the Rice Institute of America in 1915:—"There are two kinds of Mathematical Physics. Through ancient habit we regard them as belonging to a single branch and generally teach them in the same courses, but their natures are quite different. In most cases the people who are greatly interested in one despise somewhat the other. The first kind consists in a difficult and subtle analysis connected with physical questions. Its scope is to solve in a complete and exact manner the problems which it presents to us. It endeavours also to demonstrate by rigorous methods statements which are fundamental, from mathematical and logical points of view.

"I believe I do not err when I say that many physicists look upon this mathematical flora as a collection of parasitic plants grown to the great tree of natural philosophy. But this disdain is not justified. In the evolution of Mathematical Physics these researches are to play an ever increasing part........

"The other kind of Mathematical Physics has a less analytical character, but forms a subject insuperable from any consideration of phenomena. We could expect no progress in their study without the aid which this brings to them. Could any one imagine the electro-magnetic theory of light, the experiments of Hertz and wireless telegraphy, without the mathematical analysis of Maxwell which was responsible for their birth?"

To-day I shall confine myself to the first kind of Mathematical Physics and the lecture will be mainly destructive in character. But to-morrow I shall speak on the constructive side of things, when I shall take up the second kind of Mathematical Physics.

The following are the most important equations of gravitational attractions, Electrostatics and Magnetostatics with the respective years of their discovery :—

(1) $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$ (Laplace's equation, 1781)

(2) $\frac{\partial V}{\partial n_1}+\frac{\partial V}{\partial n_2}=-4\pi\sigma,$ (Equation of Coulomb and Poisson, 1811)

σ being the surface density.

(3) $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-4\pi\rho,$ (Poisson's equation, 1813)

ρ being the volume density.

(4) $W_{+0}-W_{-0}=-4\pi\sigma,$ (Helmholtz's equation, 1853)

σ being the strength.

[The exact form of the equation (2) was given by Poisson in 1811. The last equation gives the potential difference at the surface of a double layer of strength σ.]

Let us take up Poisson's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-4\pi\rho.$$

The proof as given in text-books postulates ρ to be a constant throughout a sufficiently small sphere round (x, y, z). Let u and U denote respectively the potential due to the small sphere round (x, y, z) and that due to the remaining mass. Then

$$V=U+u.$$

By Laplace's equation $\nabla^2 U=0$

and consequently $\nabla^2 V=\nabla^2 u.$

Taking ρ to be constant

$$u=2\pi\rho\,(a^2-\tfrac{1}{3}r^2)$$

at a point (ξ, η, ζ) in the sphere distant r from (x, y, z), a being the radius of the sphere. Differentiating two times with respect to ξ, η, ζ we have at the centre

$$\nabla^2 u=-\frac{2\pi\rho}{3}(2+2+2)=-4\pi\rho.$$

In 1828, Green gave a proof more or less like the above. The first really rigorous proof was given by Gauss in 1840. Gauss's proof, which I will reproduce presently, is based on the supposition that ρ is finite and continuous together with its three partial

differential coefficients of the first order. In 1882, Professor Hölder succeeded in establishing Poisson's equation on the supposition that

$$\left| \rho(\xi, \eta, \zeta) - \rho(x, y, z) \right| < Ar^{\mu}$$

where μ and A are positive constants and r is the distance between (ξ, η, ζ) and (x, y, z). The next investigator to impose still less restrictions on ρ was G. Morera who proved in 1887 that, if

$$\int_0^r \left\{ \rho(\xi, \eta, \zeta) - \rho(x, y, z) \right\} \frac{dr}{r}$$

is finite, then Poisson's equation is valid. In 1891, Kronecker attempted to establish the equation with lesser restrictions than the preceding investigators. From 1891 to 1900 practically nothing was done to improve the proof of Poisson's equation. In 1900 and the next ten years, Dr. Henrik Petrini tried to make the proof as little restrictive as possible. Some of my papers deal with the criticism of Petrini's work. Petrini's chief result is that if

$$\lim_{h=0} \int_0^{2\pi} d\phi \int_{-1}^{1} (3\mu^2 - 1)\, d\mu \int_h^a \frac{\rho dr}{r}$$

exists and is finite, then $\dfrac{\partial^2 u}{\partial x^2}$ exists and is finite, μ standing for $\cos\theta$ where θ is the angle between the radius vector and the axis of x; with similar conditions for $\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial^2 u}{\partial z^2}$; the additional condition for the validity of Poisson's equation being the continuity of ρ which condition, however, has not been sufficiently emphasized by Petrini. But cases may occur in which ρ is discontinuous. For example, I pointed out in a paper published in the *Philosophical Magazine* in 1917 that, in the case in which ρ is $\cos\left(\log \dfrac{1}{r}\right)$, Poisson's equation is invalid. Instead of $\cos\left(\log \dfrac{1}{r}\right)$ we may take, without making any material difference, $\cos\left(\log \dfrac{1}{r}\right)$ plus any constant or differentiable function. If, then, you take a solid sphere of density $2 + \cos\left(\log \dfrac{1}{r}\right)$, say, and suppose it divided into thin concentric shells, then, by Newton's theorem, the attraction due to a shell is zero at a point inside it and therefore

$\frac{\partial u}{\partial r}=0$ at the centre. The mass of the shells between the centre and the point (ξ, η, ζ) is

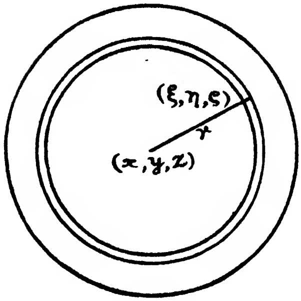

FIG. 6.

$$4\pi\int_0^r\left\{2+\cos\left(\log\frac{1}{t}\right)\right\}t^2\,dt.$$

Since the attraction at (ξ, η, ζ) is the same as if the whole of this mass were concentrated at the centre, the attraction is

$$-\frac{4\pi}{r^2}\int_0^r\left\{2+\cos\left(\log\frac{1}{t}\right)\right\}t^2\,dt$$

and is along the radius vector. Now at the centre

$$\frac{\partial^2 u}{\partial r^2}=\lim_{r=0}\frac{\frac{\partial u}{\partial r}-0}{r}=\lim_{r=0}\left[-\frac{4\pi}{r^3}\int_0^r\left\{2+\cos\log\frac{1}{t}\right\}t^2\,dt\right]$$

$$=\lim_{r=0}\left[-\frac{8\pi}{3}-\frac{4\pi}{\sqrt{10}}\left\{\cos\left(\log\frac{1}{r}+\tan^{-1}\frac{1}{3}\right)\right\}\right].$$

As r is indefinitely diminished, $\frac{1}{r}$ tends to ∞ and $\cos\left(\log\frac{1}{r}\right)$ makes an infinite number of oscillations. Therefore $\frac{\partial^2 u}{\partial r^2}$ is non-existent at the centre. Moreover the radius vector may be taken in any direction. Hence, at the centre,

$$\frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial y^2},\ \frac{\partial^2 u}{\partial z^2}$$

are all non-existent and consequently the equation of Poisson is meaningless.

For the sake of its historical importance I proceed to give you Gauss's proof of Poisson's equation :—

Let $\rho = f(\xi, \eta, \zeta)$. Then the x-component of the attraction at the point (x, y, z) is

$$X(x, y, z) = \int_T \frac{(\xi - x)\, f(\xi, \eta, \zeta) dt}{\{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{\frac{3}{2}}} \qquad (1)$$

where dt is the element of volume of the sphere and the integral is taken over the whole sphere.

Also

$$X(x + e, y, z) = \int_T \frac{(\xi - x - e)\, f(\xi, \eta, \zeta)\, dt}{\{(\xi - x - e)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{\frac{3}{2}}} \qquad (2)$$

the integral being taken throughout T, the original volume occupied by the sphere. Suppose the sphere to be displaced a little to the left so that its centre is at P_1, i. e. $(-e, 0, 0)$. Let T_1 denote the volume occupied by the sphere in its new position. Then, as is clear from the accompanying figure (7),

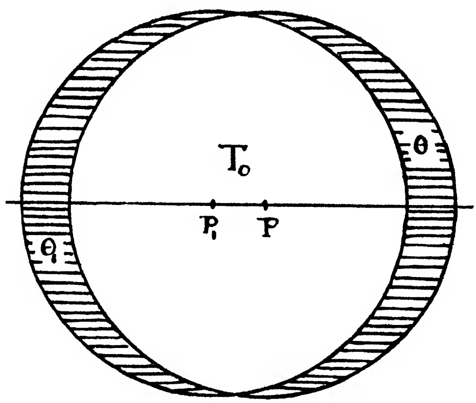

FIG. 7.

$$T - \theta = T_1 - \theta_1 = T_0.$$

Now, put

$$\xi' = \xi - e.$$

Then

$$X(x + e, y, z) = \int_{T_1} \frac{(\xi' - x)\, f(\xi' + e, \eta, \zeta)\, dt}{\{(\xi' - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{\frac{3}{2}}} \qquad (3)$$

Also

$$-\frac{X(x+e,\,y,\,z)-X(x,\,y,\,z)}{e}=\frac{1}{e}\left\{(l+\lambda)-(l_1+\lambda_1)\right\}=\frac{l-l_1}{e}+\frac{\lambda-\lambda_1}{e},$$

where l is the integral in (1) taken over T_0, λ the same integral taken over θ, l_1 the integral in (3) taken over T_0 and λ_1 the same integral taken over θ_1.

The value of an element of volume in θ and that of an element in θ_1 are respectively $e \cos \alpha \, dS$ and $-e \cos \alpha \, dS$ where α is the angle made by the external normal with the x-axis. Therefore in the limit,

$$\frac{\lambda-\lambda_1}{e}=\int_S (\xi-x)\, f(\xi,\,\eta,\,\zeta)\,.\,\frac{\xi-x}{r^4}\, dS$$

taken throughout the surface S of T_0;
also in the limit

$$\frac{l_1-l}{e}=\int \frac{\xi-x}{r^3}\,\frac{\partial\rho}{\partial\xi}\,dt$$

taken throughout T_0.

Therefore

$$\frac{\partial X}{\partial x}=\frac{\partial^2 u}{\partial x^2}=\int_{T_0}\frac{\xi-x}{r^3}\,\frac{\partial\rho}{\partial\xi}\,dt-\int_S\frac{(\xi-x)^2}{r^4}\,\rho dS,$$

and similarly for

$$\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial^2 u}{\partial z^2}.$$

Therefore

$$\nabla^2 u=\int_{T_0}\frac{1}{r^2}\,\frac{\partial\rho}{\partial r}\,dt-\int_S\frac{\rho}{r^2}\,dS$$

Since

$$dt=r^2 \sin\theta d\theta d\phi dr,$$

the volume integral in the above equality will tend to zero as the radius of the sphere is made to tend to zero; and similarly

$$-\int\frac{\rho}{r^2}\,dS \quad \text{will approach } -4\pi f\,(x,\,y,\,z).$$

Thus the equation

$$\nabla^2 V=-4\pi\rho$$

holds.

I proceed now to consider the generalization of Poisson's equation given by Dr. Petrini. Dr. Petrini was the first to note

clearly that the continuity of ρ was not ***sufficient*** for the validity of Poisson's equation. Taking $\rho = \dfrac{\cos^2 \theta}{\log \frac{1}{r}}$, he showed[1] that Poisson's equation fails because $\dfrac{\partial^2 V}{\partial x^2}$ is $+\infty$ while $\dfrac{\partial^2 V}{\partial y^2}$ and $\dfrac{\partial^2 V}{\partial z^2}$ are both equal to $-\infty$. Dr. Petrini was thus led to the following generalization :—

$$\lim_{\substack{h_1=0,\\ h_2=0,\\ h_3=0}} \left[\frac{1}{h_1} \left\{ \frac{\partial V(x+h_1, y, z)}{\partial x} - \frac{\partial V(x, y, z)}{\partial x} \right\} + \frac{1}{h_2} \left\{ \frac{\partial V(x, y+h_2, z)}{\partial y} - \frac{\partial V(x, y, z)}{\partial y} \right\} + \frac{1}{h_3} \left\{ \frac{\partial V(x, y. z+h_3)}{\partial z} - \frac{\partial V(x, y, z)}{\partial z} \right\} \right] = -4\pi\rho,$$

provided that while h_1, h_2, h_3 are made to tend to zero in any manner, $\dfrac{h_1}{h_2}$, $\dfrac{h_1}{h_3}$, $\dfrac{h_2}{h_3}$ should tend neither to zero nor to infinity. Dr. Petrini proved the validity of his generalization for $\rho = \dfrac{\cos^2 \theta}{\log \frac{1}{r}}$ and, indeed, for every case where ρ is continuous. But he did not consider the cases in which ρ has a discontinuity of the second kind. Such cases have been studied by me in the paper referred to above. I have proved, for example, that when

$$\rho = \cos\left(\log \frac{1}{r}\right),$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = -\frac{4\pi}{\sqrt{10}} \left\{ \cos\left(\log \frac{1}{r} + \tan^{-1} \frac{1}{3}\right) \right\}$$

which shows that the limit in the generalization of Petrini will exist and be equal to the limit of

$$\frac{-4\pi}{\sqrt{10}} \left\{ \cos\left(\log \frac{1}{h_1} + \tan^{-1}\frac{1}{3}\right) + \cos\left(\log \frac{1}{h_2} + \tan^{-1}\frac{1}{3}\right) + \cos\left(\log \frac{1}{h_3} + \tan^{-1}\frac{1}{3}\right) \right\}$$

if the latter limit exists. Now put $\alpha h_1 = h_2$ and $\beta h_1 = h_3$. Then denoting $\log \dfrac{1}{h_1} + \tan^{-1} \dfrac{1}{3}$ by $\log H$, the expression within the crooked brackets becomes

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[1] For another method than Petrini's, see my paper in the *Phil. Mag.*, Vol. 34, 1917.

$$\cos(\log H) + \cos\left(\log H + \log\frac{1}{\alpha}\right) + \cos\left(\log H + \log\frac{1}{\beta}\right),$$

i. e., $$\cos(\log H)\left\{1 + \cos\left(\log\frac{1}{\alpha}\right) + \cos\left(\log\frac{1}{\beta}\right)\right\}$$
$$- \sin(\log H)\left\{\sin\left(\log\frac{1}{\alpha}\right) + \sin\left(\log\frac{1}{\beta}\right)\right\}.$$

Now the only way in which this expression can have a limit is by the vanishing of both

$$\left\{1 + \cos\left(\log\frac{1}{\alpha}\right) + \cos\left(\log\frac{1}{\beta}\right)\right\} \text{ and } \left\{\sin\left(\log\frac{1}{\alpha}\right) + \sin\left(\log\frac{1}{\beta}\right)\right\}$$

which is possible only when α and β have respectively the forms

$$e^{2m\pi \pm 2\pi/3} \text{ and } e^{2n\pi \mp 2\pi/3},$$

m and n being integers. Thus, it is proved that the generalization of Petrini does not hold for every discontinuous distribution.

Another generalization of Poisson's equation was given by Professor C. W. Oseen of Upsala. As I have already taken a good deal of your time this evening, I will take up the consideration of that generalization to-morrow. I will conclude the lecture by discussing briefly the equation of Coulomb and Poisson and Helmholtz's equation.

I have studied the equation of Coulomb and Poisson in three papers one of which appeared in the *Rendiconti del Circolo Matematico di Palermo* in 1917, another in the *Philosophical Magazine* in 1918 and the third in the *Bulletin of the Calcutta Mathematical Society* in 1919. If round P, any point on the surface, a small disc be described lying on the surface, then for investigating the validity of the equation

$$\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2} = -4\pi\sigma,$$

we need only take into consideration the part of V due to the disc.

Among other results, I have proved that, denoting by r the distance of any point of the disc from P, if

$$\sigma = \cos\chi(r) \text{ when } \chi(r) \succ 0$$

and, further, if

$$\chi(r) \succsim \log\frac{1}{r}$$

then $\frac{\partial V}{\partial n_1}$, $\frac{\partial V}{\partial n_2}$ are both non-existent and the equation of Coulomb and Poisson fails.

Results similar to those mentioned above have been obtained by me for the validity of Helmholtz's equation in a paper which

appeared in the *Proceedings of the Benares Mathematical Society* for 1920. I have shown, for example, that, if the strength

$$\sigma = \cos \log \frac{1}{r},$$

then W_{+0} and W_{-0} are both non-existent and therefore the equation fails.

FIFTH LECTURE.

[Delivered on the 4th February, 1921.]

THE PARTIAL DIFFERENTIAL EQUATIONS OF THE THEORY OF GRAVITATIONAL ATTRACTIONS, AND THOSE OF ELECTROSTATICS AND MAGNETOSTATICS (*Continued*).

Yesterday I promised to lay before you the generalization of Poisson's equation given by Professor C. W. Oseen. The generalization[1] is fairly simple and will prove as useful as Poisson's equation. It is this :—

$$\int_S \frac{\partial V}{\partial n}\, ds = -4\pi \int_{\Omega_S} \rho dv,$$

where dv is an element of the volume Ω_S, ds an element of the surface S which bounds the volume Ω_S and $\frac{\partial V}{\partial n}$ is the differential coefficient with respect to the normal taken outwards. The only condition imposed here on ρ is that it should be finite and integrable.

It will be remembered that in proving the theorem

$$\nabla^2 V = -4\pi\rho$$

we have to impose more restrictions on ρ than that it should be finite and integrable ; for example, in Gauss's proof, it is supposed that ρ is continuous and also that ρ_x, ρ_y, ρ_z exist and are continuous. We see at once how much more simple Professor Oseen's generalization is.

Starting from the equation, known in Integral Calculus as Green's theorem,

$$\int (U \nabla^2 V - V \nabla^2 U) dv = \int \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) ds,$$

one can deduce the modified theorem of Green, viz.,

$$V(P) = \int_{\Omega_S} \frac{\rho'}{r}\, dv' + \frac{1}{4\pi} \int_S \left\{ \frac{1}{r} \frac{\partial V'}{\partial n} - V' \frac{\partial \left(\frac{1}{r}\right)}{\partial n} \right\} ds,$$

[1] See Oseen's paper, "Ueber einen Satz von Green und ueber die Definitionen von rot und div" (*Rendiconti del Circolo matematico di Palermo*, t. 38, 1913).

where ρ' stands for $\rho(P')$ and V' for $V(P')$, P' being the variable point (ξ, η, ζ), P the fixed point (x, y, z), and r the distance PP'.

This result follows from Green's theorem which holds when $\nabla^2 U$, $\nabla^2 V$ are existent and integrable. To show this, I take U to be $\frac{1}{PP'}$. Now $\frac{1}{PP'}$ becomes infinite at the point P. In order to make Green's theorem applicable, we enclose the point P within a small sphere Ω' of radius a. Inside the region bounded by this sphere and S, U and its differential coefficients are finite and continuous. Hence, since $\frac{1}{r}$ satisfies Laplace's equation, Green's theorem applied to this region gives

$$\int_{\Omega_S} \frac{1}{r} \nabla^2 V' dv - \int_{\Omega'} \frac{1}{r} \nabla^2 V' dv$$

$$= \int_S \left\{ \frac{1}{r} \frac{\partial V'}{\partial n} - V' \cdot \frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} ds - \int_{S'} \left\{ \frac{1}{r} \frac{\partial V'}{\partial r} - V' \cdot \frac{\partial\left(\frac{1}{r}\right)}{\partial r} \right\} ds', \qquad (1)$$

where ds' is an element of the surface S' of the sphere Ω'. Now let a tend to zero, then, dv being of the order $r^2 dr$, the second term in the left-hand side of the above equation is negligible. Similarly, as ds' is of the same order as r^2, the third term in the right-hand side is negligible. Also $\nabla^2 V' = -4\pi\rho'$ according to Poisson's equation. Hence we have in the place of (1),

$$-4\pi \int_{\Omega_S} \frac{\rho'}{r} dv = \int_S \left\{ \frac{1}{r} \frac{\partial V'}{\partial n} - V' \frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} ds + \int_{S'} V' \frac{\partial\left(\frac{1}{r}\right)}{\partial r} ds',$$

which becomes, in the limit when a tends to zero,

$$V(P) = \int_{\Omega_S} \frac{\rho'}{r} dv' + \frac{1}{4\pi} \int_S \left\{ \frac{1}{r} \frac{\partial V'}{\partial n} - V' \frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} ds .$$

This result is the modified theorem of Green and it says that the value of V at any point is the potential due to (1) a distribution of volume density ρ', (2) a certain distribution on the bounding surface and (3) a certain double layer on that surface. Thus we see that if there is a solid with a known volume distribution and prescribed values of V and $\frac{\partial V}{\partial n}$ on the bounding surface, then by applying the modified theorem of Green we can at once put down its potential V at any point.

The question arises: Can we deduce the modified theorem of

Green by using Oseen's generalization *instead* of Poisson's equation ? The question has been answered in the affirmative by Oseen who has deduced the modified theorem of Green by applying vector analysis and by employing the following theorem :—

If ϕ and ψ are two functions, finite and continuous together with their first differential coefficients, then

$$\int_{\Omega_S} (\text{grad}\ \phi \times \text{grad}\ \psi)\, dv - \int_S \psi \frac{\partial \phi}{\partial n}\, ds = 4\pi \int \psi \rho dv .$$

The procedure adopted by Oseen is rather complicated, since the second differential coefficients of ϕ may be non-existent, but the deduction is an epoch-making result.

The equation of Laplace, which I have frequently mentioned in some of my previous lectures is intimately associated with problems in Electrostatics and Magnetostatics. Thus we come across this equation when considering the case of a solenoidal distribution of electricity and the most important problem associated with the equation is Dirichlet's problem, viz. to find the potential at points inside a region when the value of the potential is given on the boundary.

With reference to Dirichlet's problem, two theorems have been formulated and proved under certain general conditions. These theorems are :—

(1) (The Theorem of Uniqueness) : The solution of Dirichlet's problem is unique.

(2) (The Existence Theorem) : The prescribed value on the boundary is arbitrary and corresponding to each value there exists a solution.

That both the theorems hold, was at first guessed from certain physical considerations. For example, in the case of a conductor connected with the earth and influenced by external electric masses. the value of the potential of the induced distribution on the surface is prescribed, being equal in magnitude, and opposite in sign, to that due to the masses, and the uniqueness and existence of the potential is obvious to a physicist whatever the masses may be. But attempts were made to base the two theorems on something more rigorous.

Dirichlet formulated the principle, known as Dirichlet's principle, that, of all the functions, V, which inside the region bounded by a closed surface remain continuous together with their first differential coefficients and take a prescribed value on the surface. there must exist one (or more) for which the integral

$$\int \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right\} dv$$

taken over the whole region has a minimum. Accepting this principle, we get from the calculus of variations the result that V must satisfy Laplace's equation.

But Weierstrass pointed out in 1860 that Dirichlet's principle was not always valid and that all that could be said was that the integral has a lower limit and that it may *not* have a minimum. In 1899, Hilbert established the validity of Dirichlet's principle on the supposition that the prescribed boundary value is continuous together with the differential coefficients of the first two orders.

Assuming that in a given case the solution of Dirichlet's problem exists and is unique, the question arises of actually obtaining this solution. I propose to devote the remaining time to-day to the discussion of a number of methods. These are (1) the use of Green's function, (2) the use of fundamental functions based on a specially suited co-ordinate system, (3) the *méthode du balayage*, (4) the method of arithmetic means of Poincaré and (5) the use of integral equations.

Adopting the procedure used in deducing the modified theorem of Green from Green's theorem but taking U to be $G+\frac{1}{r}$ instead of $\frac{1}{r}$, we obtain for $V(P)$ the simple expression

$$-\frac{1}{4\pi}\int_S V\frac{\partial U}{\partial n}\,ds;$$

here G is regular everywhere inside the region, where it satisfies Laplace's equation, and equals $-\frac{1}{r}$ at every point on the surface; $G+\frac{1}{r}$ is called Green's function for the particular region. The problem of obtaining the solution of Dirichlet's problem has been thus reduced to that of obtaining Green's function which may be interpreted as the potential due to a unit of electricity at P plus the potential due to the distribution induced on S by this unit when S is connected with the earth. In some cases Green's function is easily obtained by applying Lord Kelvin's theory of images. For example, let S be a spherical surface of radius a. Then the value of Green's function at any point Q is

$$-\frac{a}{R}\cdot\frac{1}{P'Q}+\frac{1}{PQ},$$

where R denotes the distance of P from the centre of the sphere,

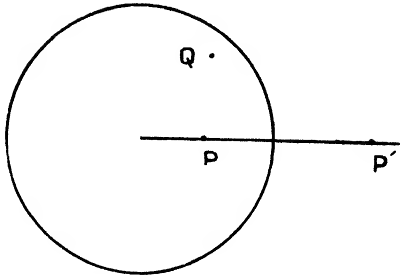

FIG. 8.

P' the inverse of P with respect to the sphere; in fact $\frac{a}{R}$ units at P' form the image of a unit at P.

I proceed now to consider at some length the second method, viz. that in which use is made of fundamental functions based on a specially selected system of co-ordinates suitable for the particular boundary surface.

As an example of this method, consider the case in which the bounding surface is a sphere. Then, transforming $\nabla^2 V=0$ into the polar co-ordinates r, θ, ϕ, $r=a$ being the bounding spherical surface, we have

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0\,.$$

We start by obtaining a typical solution of this equation in the form of a product of three factors, one a function of r alone, another a function of θ alone and the third a function of ϕ alone. Such a solution which is the desired fundamental function, is easily found to be

$$r^n P_n^m(\cos\theta)\ \begin{matrix}\cos m\phi\\ \text{or}\\ \sin m\phi\end{matrix}$$

where

$$P_n^m=\frac{1}{2^n n!}\left(1-\mu^2\right)^{\frac{m}{2}}\frac{d^{n+m}}{d\mu^{n+m}}\left(\mu^2-1\right)^n.$$

The solution of Dirichlet's problem is given by

$$V=\sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n P_n^m\left(A_{n,m}\cos m\phi+B_{n,m}\sin m\phi\right),$$

where the A's and B's are constants which have to be determined by the condition that on the boundary, $r=a$, V takes the prescribed value, say $f(\theta,\phi)$. Now in actual practice the determination of

these constants is not an easy affair. Gauss had given[1] a method for expanding any arbitrary polynomial (in the co-ordinates of a point on the spherical surface) in a series of spherical harmonics. If $\phi(x, y, z)$ be the polynomial of degree n in x, y, z where

$$x^2+y^2+z^2=a^2,$$

then Gauss put

$$\phi(x, y, z)=S_n+r^2 S_{n-2}+r^4 S_{n-4}+\ldots,$$

S_m standing for a solid spherical harmonic of degree m, and, by operating on the above equation with ∇^2, ∇^4, etc., obtained a number of simultaneous equations by solving which the S's were obtained. But no compact and general formula for writing down the expansion had been obtained until about twenty years ago I gave it in a paper published in the *Messenger of Mathematics*, Vol. 30, in 1900.

The result is

$$\phi(x, y, z)=(-1)^n\sum_{\mu=0}^{\infty}\frac{(2n-4\mu+1)\left[r^{2n-2\mu+1}\left\{\nabla_a^{2\mu}\,\phi(\alpha, \beta, \gamma)\right\}\frac{1}{r}\right]}{2\mu\,(2\mu-2)\ldots 2\,.\,(2n-2\mu+1)\,(2n-2\mu-1)\ldots 3\,.\,1}$$

where

$$\alpha=\frac{\partial}{\partial x},\quad \beta=\frac{\partial}{\partial y},\quad \gamma=\frac{\partial}{\partial z},\quad \nabla_a^2=\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}+\frac{\partial^2}{\partial\gamma^2}.$$

The above-mentioned result was obtained by me at first by a method different from that of Gauss whose method was applied by me to get the result in a paper published in the *Mathematische Annalen*, Vol. 75, in 1912.

If $f(\theta, \phi)$ is expressed as a function of x, y, z, the co-ordinates of a point on the boundary, that function can, in its turn, be expressed as a sum of polynomials each of which may be expanded by my formula, and thus an expansion for $f(\theta, \phi)$ can be obtained. With the help of this expansion, the solution of Dirichlet's problem can be put down at once as was done by me in my paper of 1900.

As a second example of the method of fundamental functions, consider the case in which the boundary is a cylinder. Then, transforming $\nabla^2 V=0$ into cylindrical co-ordinates ρ, z, ϕ, $\rho=a$ being the boundary, we get

$$\frac{\partial^2 V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

The typical solution of the above is

$$e^{\pm kz}\,J_m(k\rho)\begin{matrix}\cos m\phi\\ \text{or}\\ \sin m\phi\end{matrix};$$

[1] See his "Gesammelte Werke", Bd. 5, p. 360.

this is the desired fundamental function, and the solution of Dirichlet's problem is given by

$$V=\Sigma\left(A_{m,n}\ \cos m\phi+B_{n,m}\ \sin m\phi\right)\left(C_{n,m}\ e^{k_nz}+D_{n,m}\ e^{-k_nz}\right)J_m\left(k_n\rho\right),$$

where the constants, the A's, B's, C's and D's are determined by the boundary condition.

As the third example of the method of fundamental functions consider the case in which the boundary is an ellipsoid. Then, transforming $\nabla^2 V=0$ into the ellipsoidal co-ordinates λ, μ, ν, $\lambda=0$ being the given ellipsoidal boundary

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ .$$

we get

$$(\nu-\mu)\left\{\left(R_\lambda\frac{\partial}{\partial\lambda}\right)^2 V\right\}+(\lambda-\nu)\left\{\left(R_\mu\frac{\partial}{\partial\mu}\right)^2 V\right\}+(\mu-\lambda)\left\{\left(R_\nu\frac{\partial}{\partial\nu}\right)^2 V\right\}=0,$$

where

$$R_\theta=\sqrt{(a^2+\theta)\,(b^2+\theta)\,(c^2+\theta)}\ .$$

We suppose, as usual, that a typical solution of the above equation can be obtained as a product of three factors, each being a function of one co-ordinate alone. Thus we get as the fundamental function

$$E(\lambda)\ E(\mu)\ E(\nu),$$

where

$$\left(R_\theta\ \frac{d}{d\theta}\right)^2 E(\theta)=(m\theta+n)\ E(\theta)\ ,$$

which is Lamé's equation and can be solved in series The required solution of Dirichlet's problem is given by

$$V=\Sigma\ A_{m,n}\ E_{m,n}\ (\lambda)\ E_{m,n}\ (\mu)\ E_{m,n}\ (\nu)\ ,$$

the A's being constants which are determined by the condition that, for $\lambda=0$, V has the prescribed value, say $F(\mu,\nu)$. Many attempts have been made to obtain a compact and general formula for expanding $F(\mu,\nu)$ in ellipsoidal harmonics similar to that given by me for expansion in spherical harmonics, but so far the attempts have proved to be unsuccessful.

The question of the determination of the most general system of three orthogonal surfaces, namely, that of the confocal cyclides, for which the fundamental function is the product of three factors, as in the cases considered above, was investigated by Professor Klein and his pupil, the late Professor Maxime Bôcher. (See

Bôcher's book, "Reihenentwicklungen der Potentialtheorie", Leipzig, 1894.) For the case of an arbitrary bounding surface, Poincaré established in 1896 the existence of fundamental functions, which, however, are not necessarily expressible as the product of three factors, each factor being a function of only one coordinate. (See his paper in the *Acta Mathematica*, Vol. 20.)

Unlike the previous method, the *méthode du balayage* of Poincaré[1] uses approximation functions each of which has the prescribed value on the bounding surface and each satisfies Laplace's equation more approximately than the one preceding it. Briefly, the method may be outlined as follows :—

Let K be a sphere which encloses the surface S and the region Ω_S bounded by it and let the prescribed value $f(x, y, z)$ be such that f together with its differential coefficients of the first two orders is continuous in K; also assume that $\nabla^2 f \leqq 0$ in K, and let ϕ denote

$$-\frac{1}{4\pi}\int_K \frac{1}{r} \cdot \nabla^2 f \cdot dt \, .$$

Then obviously $\nabla^2(\phi - f) = 0$ in Ω_S. Now construct an enumerable aggregate of spheres $K_0, K_1, K_2, \ldots$ such that each sphere lies in Ω_S and fills it in such a manner that each point of Ω_S is inside at least one sphere and the spheres become indefinitely large in number only in the neighbourhood of S. Then, if the sequence of functions $U_0, U_1, U_2, \ldots$ is determined by the conditions that (1) $\nabla^2 U_0 = 0$ in K_0 and U_0 equals ϕ in $K - K_0$, (2) $\nabla^2 U_1 = 0$ in K_1 and U_1 equals U_0 in $K - K_1$, (3) $\nabla^2 U_2 = 0$ in K_2 and U_2 equals U_1 in $K - K_2$, etc., $U_0 \geqq U_1 \geqq U_2 \ldots$ and the sequence $U_0, U_1, U_2, \ldots$ converges uniformly to a regular potential-function in Ω_S: the required solution of Dirichlet's problem is

$$V = \lim_{n=\infty} U_n - \phi + f \, .$$

The case in which $\nabla^2 f > 0$ can be dealt with by making it dependent on the case considered above by suitable superposition.

The method of arithmetic means as given by Neumann[2] postulates the convexity of S and determines the solution of Dirichlet's problem as the potential of a double layer. If

$$W = -\frac{1}{2\pi}\int_S f \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) . \, ds, \quad W_1 = -\frac{1}{2\pi}\int_S W \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds \; ,$$

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[1] See his paper, "Sur les equations aux dérivées partielles de la Physique Mathématique" (*American Journal of Mathematics*, Vol. 12, 1890).

[2] See his booklet, "Untersuchungen über das logarithmische and Newtonsche Potential," Leipzig, 1877.

$$W_2 = -\frac{1}{2\pi}\int_S W_1 \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds, \qquad \text{etc.,}$$

then it has been proved by Neumann that the series

$$-C + (W - W_1) + (W_2 - W_3) + \ldots$$

$$\text{i.e.,} \quad -C - \frac{1}{2\pi}\int_S \left\{ (f - W) + (W_1 - W_2) + \ldots \right\} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds,$$

converges in Ω_S, satisfies Laplace's equation and, with the approach of the point P, at which the potential is taken, to the boundary, tends to f. Here C is a constant and equals the value of $\lim\limits_{n=\infty} W_n$ on the surface.

The method in which the theory of integral equations is used is allied to the above method inasmuch as the solution of Dirichlet's problem is obtained as the potential due to a double layer. Let the strength of the layer be ρ, then it is known that

$$W_e - W_i = -4\pi\rho(P), \qquad \text{and}$$

$$W_e + W_i = 2\int_S \rho(M)\frac{\cos\psi}{r^2} ds,$$

where W_e or W_i is the limit to which the potential W of the double layer tends according as the point P on the surface is approached from the outside or from the inside, P, M, ψ having the same meanings as in the figure on p. 6. Adding the two equations, we have

$$2W_e = -4\pi\rho(P) + 2\int_S \rho(M)\frac{\cos\psi}{r^2} ds.$$

But W_e is equal to the prescribed value $f(P)$. Therefore ρ is given by the integral equation

$$\rho(P) = -\frac{1}{2\pi} f(P) + \frac{1}{2\pi}\int_S \rho(M)\frac{\cos\psi}{r^2} ds.$$

SIXTH LECTURE.

[Delivered on the 5th February, 1921.]

THE PARTIAL DIFFERENTIAL EQUATIONS OF HYDRODYNAMICS AND THOSE OF ELECTRODYNAMICS AND THE THEORY OF ELECTRONS.

I propose to speak to-day first on the partial differential equations of Hydrodynamics. Less than three hundred years ago, Hydrodynamics was not a science at all. In fact the first theorem ever discovered in Hydrodynamics is what is known as Torricelli's theorem. This was first stated in the year 1643. Torricelli was one of the greatest engineers of his day. He found from observation that the velocity of water issuing from a small hole in a vessel of water is the same as that of a particle falling from the level of the water in the vessel, i.e., $q^2=2gz$, where q is the velocity of water flowing out and z is the depth of the hole below the water level. The first attempt at a mathematical proof of this, was made by Newton. He pointed out that two important points were to be kept in mind. (1) The fluid particles in any horizontal section of the jet will always remain in a horizontal section. (2) As the section increases in size, the velocity in the section becomes less in the inverse ratio. This was a crude form of the equation of continuity. From considerations like these Newton satisfied himself that the theorem of Torricelli was correct. He published his arguments in the *Principia* in 1687. But the theorem was first proved completely by D. Bernoulli in 1738. In 1755 Euler gave the equations of motion

$$\frac{Du}{Dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{Dw}{Dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z},$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z},$$

u, v, w being the components of velocity and p the pressure at (x, y, z). The equation of continuity was also given by him. It was first published in the *Memoirs of the Academy of Sciences of St. Petersburg* in 1770. It is

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$$

With these equations Euler proved Torricelli's theorem in the following manner :—

When the motion is steady, we have $\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\frac{\partial w}{\partial t}=0$. Also, because the only external force is gravity, we have $X=Y=0$, and $Z=g$. The equations of motion become

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}; \qquad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y};$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=g-\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Since ρ is constant for ordinary fluids, the equation of continuity becomes $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$.

Further, since u, v, w are the component velocities in an irrotational motion, we have

$$u=\frac{\partial \phi}{\partial x},\ v=\frac{\partial \phi}{\partial y},\ w=\frac{\partial \phi}{\partial z}.$$

Hence $\frac{\partial u}{\partial y}=\frac{\partial^2 \phi}{\partial y \partial x}=\frac{\partial v}{\partial x}$. And we have two similar equations.

$$\therefore\ u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}.\quad \text{Or, } \tfrac{1}{2}\frac{\partial}{\partial x}(q^2)=-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Similarly, $\tfrac{1}{2}\frac{\partial}{\partial y}(q^2)=-\frac{1}{\rho}\frac{\partial p}{\partial y}$, and $\tfrac{1}{2}\frac{\partial}{\partial z}(q^2)=g-\frac{1}{\rho}\frac{\partial p}{\partial z}$.

Multiply these equations by dx, dy, dz respectively and integrate; then

$$\tfrac{1}{2}q^2=\frac{-p}{\rho}+gz+C;$$

which is a relation between the pressure and velocity.

Since the hole is small, we can neglect q^2 at the level of the water. This gives

$$0=\frac{-P_0}{\rho}+C.$$

$$\therefore\ \tfrac{1}{2}q^2=\frac{P_0-p}{\rho}+gz.\quad \text{But } \frac{P_0-p}{\rho}=0, \text{ at the hole;}$$

$\therefore$ at the hole, $\frac{1}{2}q^2=gz$, which proves the theorem.

It will be noticed that the equation of continuity, viz., $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ becomes, for irrotational motion, $\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0$,

which is the same as Laplace's equation about which I spoke yesterday. Problems in Hydrodynamics can therefore be treated in the same way as problems in Electrostatics or Heat.

For rotational or vortex motion, the equations of motion remain the same as before. We have also

$$2\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z};\quad 2\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x};\quad 2\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y};$$

where ξ, η, ζ are the components of the curl.

Then

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z},$$

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial v}{\partial x}+\frac{\eta}{\rho}\frac{\partial v}{\partial y}+\frac{\zeta}{\rho}\frac{\partial v}{\partial z},$$

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial w}{\partial x}+\frac{\eta}{\rho}\frac{\partial w}{\partial y}+\frac{\zeta}{\rho}\frac{\partial w}{\partial z}.$$

We can solve these equations in terms of a scalar potential Φ (no longer a velocity potential).

If $\Phi=-\iiint\frac{\theta'}{4\pi r}\,dx'\,dy'\,dz'$, $\quad F=\iiint\frac{\xi'}{4\pi r}\,dx'\,dy'\,dz'$, etc.,

r being $\quad\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$,

and $\quad\theta=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z};$

then

$$u=\frac{\partial\Phi}{\partial x}+\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z},$$

$$v=\frac{\partial\Phi}{\partial y}+\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x},$$

$$w=\frac{\partial\Phi}{\partial z}+\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}.$$

We can thus express u, v, w in terms of Φ, F, G, H. The analysis is very much the same as that which is used for problems on the motion of electricity. This is an important point of view and is of special importance in the treatment of viscous fluids.

The first attempt to study the motion of viscous fluids was made by Navier in 1827 and the investigation was extended by Poisson in 1831. But all the investigations which treated the problem from the point of view of molecular theory, took the mole-

cules as mere points, although they were supposed to be of finite size at the start. These investigations were therefore only fruitful in creating interest, but were not of much scientific importance. The first person to dea with the problem with any rigour was De St. Venant, who is famous for his work in the theory of Elasticity. The equations of motion are

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial \Theta}{\partial x} + \nu \nabla^2 u,$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\nu}{3}\frac{\partial \Theta}{\partial y} + \nu \nabla^2 v,$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{3}\frac{\partial \Theta}{\partial z} + \nu \nabla^2 w,$$

where Θ is the dilatation and equals $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.

These equations have a meaning only when the first and the second differential coefficients of u, v, w exist and are finite. The continuity of these had also been postulated, in deriving the equations. Prof. Oseen showed that the motion can be determined even in the case when $\frac{\partial p}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}$, etc., do not exist at all. He critically examined the equations, just as I have done for the conduction of heat [1] and obtained

$$\int_{-\pi}^{x} \{T(x,t) - T(x,0)\} d\tau = \int_{0}^{t} \frac{\partial T(x,\tau)}{\partial x} d\tau,$$

as the equation of the phenomenon of linear conduction.

Oseen [2] has obtained certain very interesting results. I read out to you a translation of his own views on the subject: "The differential equations of Navier admit of being transformed into integral equations in which the differential coefficients

$$\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \text{ etc., } \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z},$$

do not occur. Indeed it can be shown that the integral equations possess systems of solutions for which these differential coefficients do not at all exist. One must then ask whether these

[1] "Constitution of Matter and Analytical Theories of Heat" (*Abhandbungen der Gesellschaft der Wissenschaften zu Göttingen*. Bd. 2, 1903).

[2] "Ueber die Bedeutung der Integralgleichungen in der Theorie der Bewegung einer reibenden, unzusammendrückbaren Flussigkeit" (*Arkiv för Matematik, Astronomi och Fysik*, Bd. 6, 1910).

solutions of the integral equations, which are not at the same time solutions of the differential equations, possess only a mathematical importance or whether some among them have a physical importance. I undertake the task to show that the latter is the case and that therefore not the differential equations of Navier but the integral equations should be considered as expressing adequately the physical hypotheses which lie at the basis of the theory of the motion of a viscous fluid.

"The direct derivation of the integral equations of Hydrodynamics without the intermediation of Navier's differential equations, appears to me to be of interest from two points of view. First, obviously the theory gains thereby in generality, secondly, the theory gains much in simplicity. In order to see this, we need only consider how complicated the considerations are which will be necessary to decide under what circumstances the potential functions will possess differential coefficients of the second order. These questions and others of the same kind occur in Hydrodynamics if one starts from Navier's differential equations. If one, however, obtains directly the integral equations of Hydrodynamics all these questions lose their importance. A similar simplification is obtainable in a similar manner in some branches of mathematical Physics. I may mention as examples the theory of potential, and the theory of the conduction of Heat."

I give you now the integral equations of Oseen. They bear a resemblance to the equations for the vibrating string with an infinite number of edges.

From the principle:

Amount of motion coming into a closed volume
=amount of motion due to extraneous forces,

we get

$$\rho\int_\omega (u_{t+\Delta t} - u_t)\,dv - \int_t^{t+\Delta t} d\tau \int_\omega X'(\tau)\,dv + \int_t^{t+\Delta t} d\tau \int_S X_n(\tau)\,ds = 0\ ;$$

and two similar equations;

where $$X' = X - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right),$$

and $$X_n = X_x \cos nx + X_y \cos ny + X_z \cos nz.$$

Here X_x, X_y X_z are stresses acting in a plane perpendicular to the x-axis. In the case of a perfect fluid the stress is parallel to the axis. In that case X_y and X_z are zero. Thus in the case of viscous fluid there are nine component stresses to consider.

The differential equation has thus been replaced by an integral equation. There is no great difficulty in the solution of such

an equation and we can show that the old solution satisfies this equation.

I now come to Electrodynamics. The equations are

$$4\pi u = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z},$$

$$4\pi v = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x},$$

$$4\pi w = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y},$$

$$-\frac{da}{dt} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z},$$

$$-\frac{db}{dt} = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x},$$

$$-\frac{dc}{dt} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y},$$

where u, v, w are the components of the total current strength at (x, y, z), α, β, γ are the components of the magnetic force, a, b, c are the components of magnetic induction and X, Y, Z are the components of the electric force. The equations can be solved by the use of vectors (F, G, H) and the potential Φ (cf. rotational motion in Hydrodynamics).

The equations of the theory of electrons are more general, and include the equations given above as a particular case. Prof. H. A. Lorentz has studied the motion of electrons. The equations of motion are

$$\frac{1}{c}\frac{\partial X}{\partial t} + \rho\frac{u}{c} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z},$$

$$\frac{1}{c}\frac{\partial Y}{\partial t} + \rho\frac{v}{c} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x},$$

$$\frac{1}{c}\frac{\partial Z}{\partial t} + \rho\frac{w}{c} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y},$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \rho,$$

$$-\frac{\partial a}{\partial t} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \text{ etc., as before,}$$

where ρ is called the volume density of the electricity and u, v, w are regarded as the component velocities of the element of electricity which is at the point (x, y, z) at time t.

The method of solution is as follows :—

Put
$$\alpha=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z},$$
$$\beta=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x},$$
$$\gamma=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y},$$
$$X=-\frac{1}{c}\frac{\partial F}{\partial t}-\frac{\partial V}{\partial x},$$
$$Y=-\frac{1}{c}\frac{\partial G}{\partial t}-\frac{\partial V}{\partial y},$$
$$Z=-\frac{1}{c}\frac{\partial H}{\partial t}-\frac{\partial V}{\partial z}.$$
where, F, G, H, are solutions of
$$\nabla^2 F-\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}+\rho\frac{u}{c}=0,$$
$$\nabla^2 G-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}+\rho\frac{v}{c}=0,$$
and
$$\nabla^2 H-\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}+\rho\frac{w}{c}=0;$$
also
$$\nabla^2 V-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}+\rho=0,$$
and
$$\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}+\frac{1}{c}\frac{\partial V}{\partial t}=0.$$

There is no difficulty now in getting a solution. If $\rho=0$, we have the equation of wave propagation. From this solution as a basis we can build up the complete solution.

Partial differential equations are found to be quite serviceable for all ordinary purposes. But in a rigourous treatment of any problem they have to be relegated to a secondary place. The work of the last 120 years is of importance as it shows that most results and processes dealing with differential coefficients are valid only under certain conditions. Thus Taylor's theorem is not always valid. For example, the expansion of $e^{-\frac{1}{(x-a)^2}}$ according to this theorem, viz..
$$e^{-\frac{1}{(x-a)^2}}=\sum_0^\infty\frac{f^n(a)}{n!}(x-a)^n,$$

is wrong because every member of the expansion is zero. This discovery, due to Cauchy in 1815, astounded Lagrange who believed that if the differential coefficients of all orders existed and were finite, the function would be expansible by Taylor's theorem. But I have mentioned to you an example which is in direct contradiction to his belief. However, in spite of its limitations, the partial differential equation will continue to be useful.

Now I will briefly recapitulate the essence of what I have said in criticizing the common use of partial differential equations. If we have a string with an edge, the first and second differential coefficients do not exist at the edge. The equation of motion becomes meaningless in this case. If we have the initial state of heat

$$f(x) = x \text{ for } x > 0, \qquad \text{and } f(x) = -x \text{ for } x < 0,$$

the first and the second differential coefficients are non-existent at the origin. Consequently, the equation of the linear conduction of Heat $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ has no meaning for $t = 0$, $x = 0$. In Electrostatics, Magnetostatics or Gravitational Potential, if $\rho = \cos \log \frac{1}{r}$, $\nabla^2 V$ does not exist. In Electrodynamics too a similar remark applies. In Hydrodynamics, as Prof. Oseen says, it is not always the differential equations but the integral equations which truly express the fundamental physical hypotheses.

It is quite possible that at some time in the future we shall be so advanced in our thoughts that a differential equation will appear to us as but a very crude mathematical instrument, and we shall discard it in favour of the more powerful and more refined integral equation.

It has been a real pleasure to me, gentlemen, to come and deliver lectures before you, and I shall be much pleased if I find that my lectures have been of use in fostering a spirit of research here.

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